Thermoelastic Relations in Isotropic Elasticity and applications to axisymmetric problems

From the book: Mechanics of Continuous Media: an Introduction

1. J Botsis and M Deville, PPUR 2018
2. J Botsis, Class notes given during the course

Thermal Stresses & Strains

We have for the strains from energetic considerations (three dimensional case):

$$\boxed{\boldsymbol{\varepsilon} = \frac{1}{2\mu}\left(\boldsymbol{\sigma} + \left(2\mu\alpha(T - T_0) - \frac{\lambda}{3\lambda + 2\mu}\,\mathrm{tr}\,\boldsymbol{\sigma}\right)\boldsymbol{I}\right)}$$

Recall

$$\boxed{\lambda = \frac{Ev}{(1+v)(1-2v)}; \quad \mu = \frac{E}{2(1+v)}}$$

Setting $(\Delta T = T - T_o)$ the above relations are written explicitly:

$$\boxed{\varepsilon_{11} = \frac{1}{E}\left(\sigma_{11} - v\sigma_{22} - v\sigma_{33}\right) + \alpha\Delta T, \ldots\ldots \quad \varepsilon_{12} = \frac{1+v}{E}\sigma_{12}, \ldots\ldots}$$

Invert the relations above to obtain:

$$\boxed{\sigma_{ij} = \lambda\varepsilon_{kk}\,\delta_{ij} + 2\mu\varepsilon_{ij} - (3\lambda + 2\mu)\alpha(T - T_0)\,\delta_{ij}\,.}$$

$$\boxed{\sigma_{11} = \frac{E}{(1+v)(1-2v)}\left[(1-v)\varepsilon_{11} + v\varepsilon_{22} + v\varepsilon_{33}\right] - \frac{E}{(1-2v)}\alpha\Delta T, \ldots\ldots \quad \varepsilon_{12} = \frac{1+v}{E}\sigma_{12}, \ldots\ldots}$$

Mechanics of Solids: Temperature changes

THERMAL STRESSES & STRAINS

$$\boxed{\boldsymbol{\varepsilon} = \frac{1}{2\mu}\left(\boldsymbol{\sigma} + \left(2\mu\alpha(T - T_0) - \frac{\lambda}{3\lambda + 2\mu}\,\mathrm{tr}\,\boldsymbol{\sigma}\right)\boldsymbol{I}\right)}$$

$$\boxed{\sigma_{ij} = \lambda\varepsilon_{kk}\,\delta_{ij} + 2\mu\varepsilon_{ij} - (3\lambda + 2\mu)\alpha(T - T_0)\,\delta_{ij}}.$$

In **plane stress** we have with $(\Delta T = T - T_o)$

$$\varepsilon_{11} = \frac{1}{E}\left(\sigma_{11} - v\sigma_{22}\right) + \alpha\Delta T;$$

$$\varepsilon_{22} = \frac{1}{E}\left(\sigma_{22} - v\sigma_{22}\right) + \alpha\Delta T;$$

$$\varepsilon_{12} = \frac{1}{2G}\sigma_{12}$$

Invert

$$\sigma_{11} = \frac{E}{1 - v^2}\left(\varepsilon_{11} + v\varepsilon_{22}\right) - \frac{E\alpha\Delta T}{1 - v};$$

$$\sigma_{22} = \frac{E}{1 - v^2}\left(\varepsilon_{22} + v\varepsilon_{11}\right) - \frac{E\alpha\Delta T}{1 - v};$$

$$\sigma_{12} = 2G\varepsilon_{12}$$

Mechanics of Solids: Temperature changes

THERMAL STRAIN & STRESSES

Plane Strain

$$\varepsilon_{11} = \frac{1}{E}\left(\sigma_{11} - v\sigma_{22} - v\sigma_{33}\right) + \alpha\Delta T$$

$$\varepsilon_{22} = \frac{1}{E}\left(\sigma_{22} - v\sigma_{33} - v\sigma_{11}\right) + \alpha\Delta T$$

$$\varepsilon_{33} = \frac{1}{E}\left(\sigma_{33} - v\sigma_{11} - v\sigma_{22}\right) + \alpha\Delta T$$

$$\varepsilon_{12} = \frac{1+v}{2}\sigma_{12}, \ldots\ldots$$

$$\varepsilon_{33} = \frac{1}{E}\left(\sigma_{33} - v\sigma_{11} - v\sigma_{22}\right) + \alpha\Delta T = 0$$

$$\Rightarrow \quad \sigma_{33} = v(\sigma_{11} + \sigma_{22}) - \alpha E\Delta T$$

Insert σ_{33} in the expressions for $\varepsilon_{11,}\varepsilon_{22}$

$$\varepsilon_{11} = \frac{1+v}{E}\left[\left(1-v\right)\sigma_{11} - v\sigma_{22}\right) + \alpha E\Delta T\right]$$

$$\varepsilon_{22} = \frac{1+v}{E}\left[\left(1-v\right)\sigma_{22} - v\sigma_{11}\right) + \alpha E\Delta T\right]$$

$$\varepsilon_{12} = \frac{1}{2G}\sigma_{12}, \ldots.$$

The stress components are obtained by inverting the above relations

Mechanics of Solids: Temperature changes

BIHARMONIC EQUATION (PLANE STRESS)

Introduce the strains

$$\varepsilon_{11} = \frac{1}{E}\left(\sigma_{11} - v\sigma_{22}\right) + \alpha\Delta T;$$

$$\varepsilon_{22} = \frac{1}{E}\left(\sigma_{22} - v\sigma_{11}\right) + \alpha\Delta T;$$

$$\varepsilon_{12} = \frac{1}{2G}\sigma_{12}$$

in the compatibility equation

$$\frac{\partial^2 \varepsilon_{11}}{\partial {x_2}^2} + \frac{\partial^2 \varepsilon_{22}}{\partial {x_1}^2} = 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2}\ .$$

And combine it with the equilibrium equations with no body forces

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + f_1 = 0 \quad \frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + f_2 = 0$$

Compatibility in terms of stresses

$$\boxed{\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)\left(\sigma_{11} + \sigma_{22} + \alpha E \Delta T\right) = 0}$$

Use the definitions

$$\sigma_{11} = \frac{\partial^2 \Phi}{\partial {x_2}^2}\ ,\ \sigma_{22} = \frac{\partial^2 \Phi}{\partial {x_1}^2}\ ,\ \sigma_{12} = -\frac{\partial^2 \Phi}{\partial x_1 \partial x_2}$$

$$\boxed{\nabla^4 \Phi + \alpha E \nabla^2 (\Delta T) = 0}$$

Mechanics of Solids: Mechanics of Solids: Temperature changes

BIHARMONIC EQUATION WITH TEMPERATURE

$$\boxed{\nabla^4\Phi + \alpha E \nabla^2(\Delta T) = 0}$$

Valid for plane stress or plane strain problems if the body forces are neglected.

BIHARMONIC EQUATION IN CYLINDRICAL COORDINATES

(only r dependence)

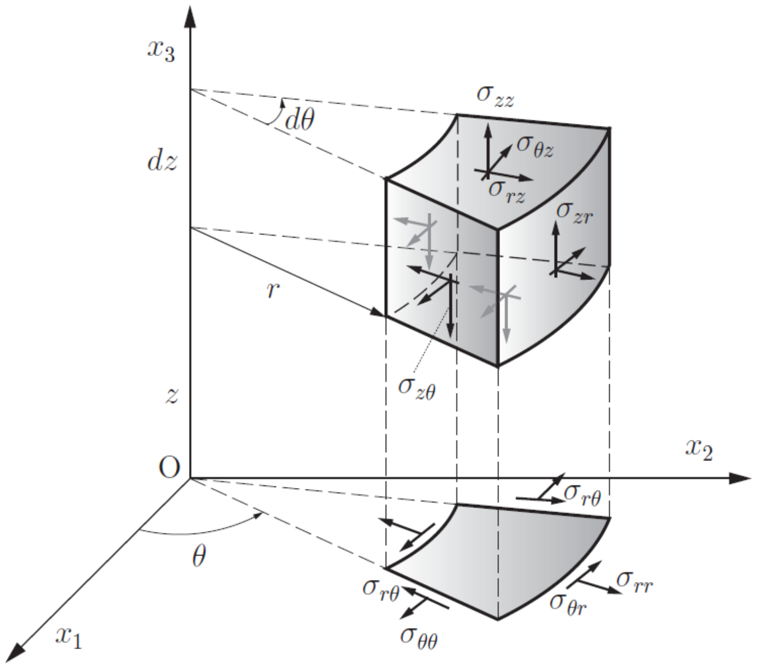
$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2\Phi}{dr^2} + \frac{1}{r}\frac{d\Phi}{dr}\right) + \alpha E(\Delta T)\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right) = 0$$

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\left[\left(\frac{d^2\Phi}{dr^2} + \frac{1}{r}\frac{d\Phi}{dr}\right) + \alpha E(\Delta T)\right] = 0$$

$$\Rightarrow \left(\frac{d^2\Phi}{dr^2} + \frac{1}{r}\frac{d\Phi}{dr}\right) + \alpha E(\Delta T) = 0$$

$$\Rightarrow \frac{1}{r}\frac{d}{dr}\left(r\frac{d\Phi}{dr}\right) + \alpha E(\Delta T) = 0$$

Mechanics of Solids: Temperature changes-cylindrical coordinates



Recall the equations

STRESS-STRAIN RELATIONS (PLANE STRESS)

(we need only replace $x_1, x_2, x_3 \rightarrow r, \theta, z$ in the corresponding relations on Cartesian coordinates)

$$\varepsilon_{rr} = \frac{1}{E}\left(\sigma_{rr} - v\sigma_{\theta\theta}\right) + \alpha\Delta T;$$

$$\varepsilon_{\theta\theta} = \frac{1}{E}\left(\sigma_{\theta\theta} - v\sigma_{rr}\right) + \alpha\Delta T;$$

$$\varepsilon_{r\theta} = \frac{1+v}{E}\sigma_{r\theta}$$

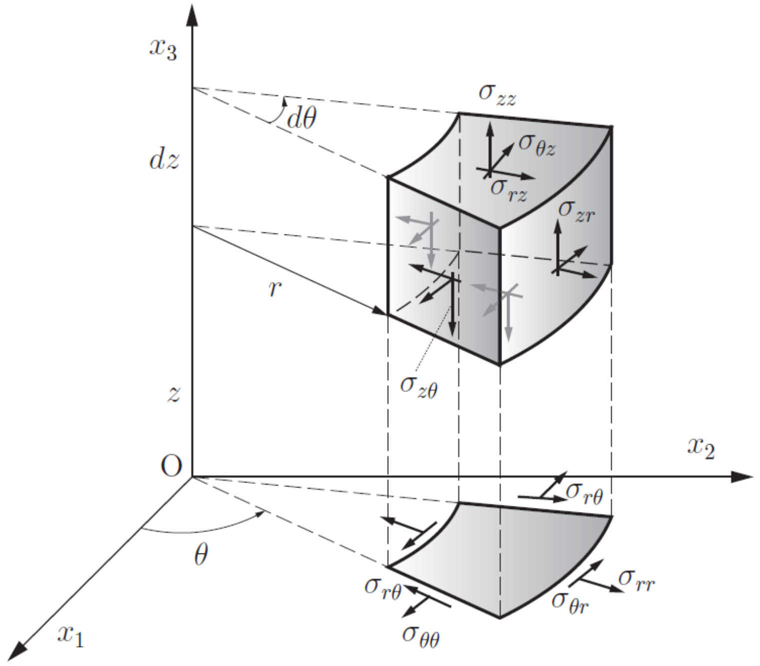
$$\sigma_{rr} = \frac{E}{1-v^2}\left(\varepsilon_{rr} + v\varepsilon_{\theta\theta}\right) - \frac{E\alpha\Delta T}{1-v};$$

$$\sigma_{\theta\theta} = \frac{E}{1-v^2}\left(\varepsilon_{\theta\theta} + v\varepsilon_{rr}\right) - \frac{E\alpha\Delta T}{1-v};$$

$$\sigma_{r\theta} = \frac{E}{(1+v)}\varepsilon_{r\theta}$$

Mechanics of Solids: Temperature changes

Thermal Stresses: Long Cylinder with fixed ends (plane strain)



STRESS-STRAIN RELATIONS (PLANE STRAIN)

(we need only replace

$$x_1, x_2, x_3 \to r, \theta, z$$

in the corresponding relations on Cartesian Coordinates)

$$\varepsilon_{rr} = \frac{1}{E}\left(\sigma_{rr} - v\sigma_{\theta\theta} - v\sigma_{zz}\right) + \alpha\Delta T$$

$$\varepsilon_{\theta\theta} = \frac{1}{E}\left(\sigma_{\theta\theta} - v\sigma_{zz} - v\sigma_{rr}\right) + \alpha\Delta T$$

$$\varepsilon_{zz} = \frac{1}{E}\left(\sigma_{zz} - v\sigma_{rr} - v\sigma_{\theta\theta}\right) + \alpha\Delta T$$

For plane strain $\varepsilon_{zz} = 0$
and $\sigma_{zz} = v(\sigma_{rr} + \sigma_{\theta\theta}) - \alpha E\Delta T$

Introduce σ_{zz} into the deformation expressions to obtain

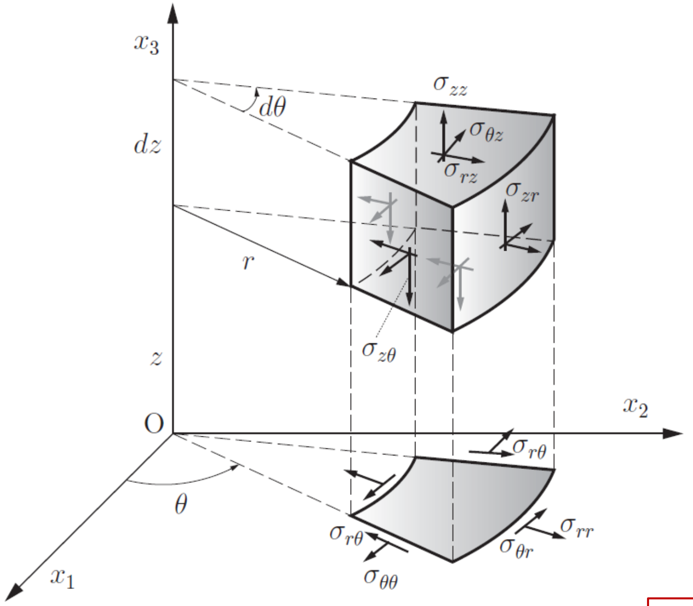
$$\varepsilon_{rr} = \frac{1+v}{E}\left[(1-v)\sigma_{rr} - v\sigma_{\theta\theta}) + \alpha E\Delta T\right]$$

$$\varepsilon_{\theta\theta} = \frac{1+v}{E}\left[(1-v)\sigma_{\theta\theta} - v\sigma_{rr}) + \alpha E\Delta T\right]$$

Mechanics of Solids: Theory of Elasticity; Eqs in cylindrical coordinates

Cylinder with free ends (plane stress)

RECALL OF THE SOLUTION METHOD



Recall the equations

$$\sigma_{rr} = \frac{E}{1-\nu^2}\left(\varepsilon_{rr} + \nu\varepsilon_{\theta\theta}\right)$$

$$\sigma_{\theta\theta} = \frac{E}{1-\nu^2}\left(\varepsilon_{\theta\theta} + \nu\varepsilon_{rr}\right)$$

$$\boxed{\varepsilon_{rr} = \frac{du_r}{dr}; \quad \varepsilon_{\theta\theta} = \frac{u_r}{r}}$$

$$\sigma_{rr} = \frac{E}{1-\nu^2}\left(\frac{du_r}{dr} + \nu\frac{u_r}{r}\right)$$

$$\sigma_{\theta\theta} = \frac{E}{1-\nu^2}\left(\frac{u_r}{r} + \nu\frac{du_r}{dr}\right)$$

$$\boxed{\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0}$$

$$\boxed{\frac{d^2u_r}{dr^2} + \frac{1}{r}\frac{du_r}{dr} - \frac{u_r}{r^2} = 0}$$

or $\boxed{\dfrac{d}{dr}\left[\dfrac{1}{r}\dfrac{d(ru_r)}{dr}\right] = 0}$ The problem is reduced to one equation with one unknown

Mechanics of Solids: Temperature changes

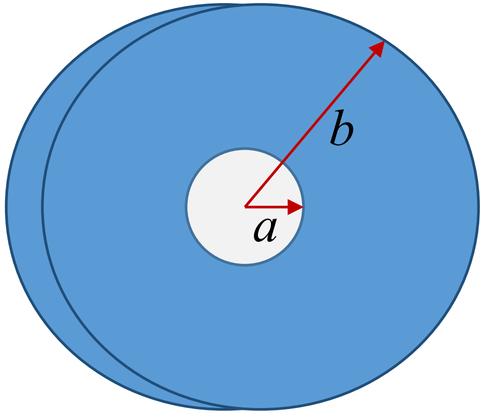
THERMAL STRESSES IN DISKS

(STATE OF PLANE STRESS)

EXAMPLES

- **ANNULAR FINS**
- **TURBINE DISKS**

The Temperature varies radially $T(r)$



Stress-strain relations

$$\varepsilon_{rr} = \frac{1}{E}\left(\sigma_{rr} - v\sigma_{\theta\theta}\right) + \alpha\Delta T;$$

$$\varepsilon_{\theta\theta} = \frac{1}{E}\left(\sigma_{\theta\theta} - v\sigma_{rr}\right) + \alpha\Delta T$$

Invert them to obtain strain-stress relations

$$\sigma_{rr} = \frac{E}{1-v^2}\left(\varepsilon_{rr} + v\varepsilon_{\theta\theta}\right) - \frac{E\alpha\Delta T}{1-v}$$

$$\sigma_{\theta\theta} = \frac{E}{1-v^2}\left(\varepsilon_{\theta\theta} + v\varepsilon_{rr}\right) - \frac{E\alpha\Delta T}{1-v}$$

Mechanics of Solids: Temperature changes

Thermal Stresses in Disks
(State of Plane stress)

Examples

- **Annular fins**
- **Turbine Disks**

The Temperature varies radially $T(r)$

Combine the strain-displacements with the strain-stress

$$\varepsilon_{rr} = \frac{du_r}{dr}; \quad \varepsilon_{\theta\theta} = \frac{u_r}{r}$$

$$\sigma_{rr} = \frac{E}{1-v^2}\left(\varepsilon_{rr} + v\varepsilon_{\theta\theta}\right) - \frac{E\alpha\Delta T}{1-v};$$

$$\sigma_{\theta\theta} = \frac{E}{1-v^2}\left(\varepsilon_{\theta\theta} + v\varepsilon_{rr}\right) - \frac{E\alpha\Delta T}{1-v};$$

Introduce the resulting expressions in equilibrium equation

$$\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0$$

To obtain:

$$\frac{d^2u_r}{dr^2} + \frac{1}{r}\frac{du_r}{dr} - \frac{u_r}{r^2} = (1+v)\alpha\frac{dT(r)}{dr}$$

(Equilibrium in terms of radial displacement)

(one equation with one unknown)

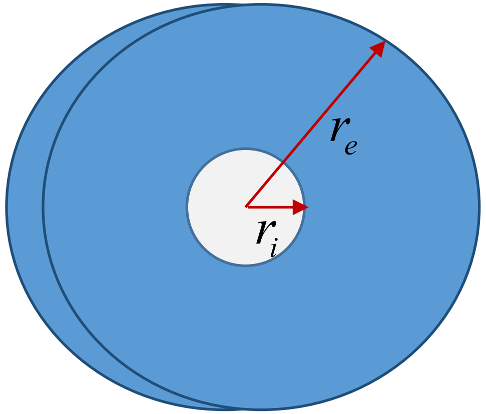
(The temperature field is given)

THERMAL STRESSES IN DISKS

EXAMPLES

- **ANNULAR FINS**
- **TURBINE DISKS**

The Temperature varies radially $T(r)$



SOLVE THE ORDINARY DIFFERENTIAL EQUATION OBTAINED EARLIER

$$\boxed{\frac{d^2u_r}{dr^2}+\frac{1}{r}\frac{du_r}{dr}-\frac{u_r}{r^2}=(1+v)\alpha\frac{dT(r)}{dr}}$$

IS REWRITTEN AS

$$\boxed{\frac{d}{dr}\left[\frac{1}{r}\frac{d(ru_r)}{dr}\right]=(1+v)\alpha\frac{dT(r)}{dr}}$$

INTEGRATE TWICE TO OBTAIN THE SOLUTION

$$\boxed{u_r=\frac{(1+v)\alpha}{r}\int_{r_i}^{r}T(r)rdr+c_1r+\frac{c_2}{r}}$$

Particular solution

Homogeneous solution

Mechanics of Solids: Temperature changes

Thermal Stresses in Disks

$$u_r = \frac{(1+\nu)\alpha}{r}\int_{r_i}^{r} T(r)rdr + c_1 r + \frac{c_2}{r}$$

$$\varepsilon_{rr} = \frac{du_r}{dr}; \quad \varepsilon_{\theta\theta} = \frac{u_r}{r}$$

$$\sigma_{rr} = \frac{E}{1-\nu^2}\left(\varepsilon_{rr} + \nu\varepsilon_{\theta\theta}\right) - \frac{E\alpha\Delta T}{1-\nu};$$

$$\sigma_{\theta\theta} = \frac{E}{1-\nu^2}\left(\varepsilon_{\theta\theta} + \nu\varepsilon_{rr}\right) - \frac{E\alpha\Delta T}{1-\nu};$$

Annular Disks

$$\sigma_{rr} = -\frac{E\alpha}{r^2}\int_{r_i}^{r} T(r)rdr + \frac{E}{1-\nu^2}\left[c_1(1+\nu) - \frac{c_2(1-\nu)}{r^2}\right]$$

$$\sigma_{\theta\theta} = \frac{E\alpha}{r^2}\int_{r_i}^{r} T(r)rdr - E\alpha T + \frac{E}{1-\nu^2}\left[c_1(1+\nu) + \frac{c_2(1-\nu)}{r^2}\right]$$

Two Boundary Conditions:

$$\sigma_{rr}\big|_{r=r_i} = \sigma_{rr}\big|_{r=r_e} = 0$$

$$c_1 = \frac{(1-\nu)\alpha}{r_e^2 - r_i^2}\int_{r_i}^{r_e} T(r)rdr; \quad c_2 = \frac{(1+\nu)r_i^2\alpha}{r_e^2 - r_i^2}\int_{r_i}^{r_e} T(r)rdr$$

$$\sigma_{rr} = E\alpha\left[-\frac{1}{r^2}\int_{r_i}^{r} T(r)rdr + \frac{r^2 - r_i^2}{r^2(r_e^2 - r_i^2)}\int_{r_i}^{r_e} T(r)rdr\right]; \quad \sigma_{\theta\theta} = E\alpha\left[-T(r) + \frac{1}{r^2}\int_{r_i}^{r} T(r)rdr + \frac{r^2 + r_i^2}{r^2(r_e^2 - r_i^2)}\int_{r_i}^{r_e} T(r)rdr\right]$$

Mechanics of Solids: Temperature changes

Thermal Stresses in Disks

$$u_r = \frac{(1+\nu)\alpha}{r}\int_{r_i}^{r} T(r)rdr + c_1 r + \frac{c_2}{r}$$

$$\varepsilon_{rr} = \frac{du_r}{dr}; \quad \varepsilon_{\theta\theta} = \frac{u_r}{r}$$

$$\sigma_{rr} = \frac{E}{1-\nu^2}\left(\varepsilon_{rr} + \nu\varepsilon_{\theta\theta}\right) - \frac{E\alpha\Delta T}{1-\nu};$$

$$\sigma_{\theta\theta} = \frac{E}{1-\nu^2}\left(\varepsilon_{\theta\theta} + \nu\varepsilon_{rr}\right) - \frac{E\alpha\Delta T}{1-\nu};$$

Solid Disks

$$\sigma_{rr} = -\frac{E\alpha}{r^2}\int_{r_i}^{r} T(r)rdr + \frac{E}{1-\nu^2}\left[c_1(1+\nu) - \frac{c_2(1-\nu)}{r^2}\right]$$

$$\sigma_{\theta\theta} = \frac{E\alpha}{r^2}\int_{r_i}^{r} T(r)rdr - E\alpha T + \frac{E}{1-\nu^2}\left[c_1(1+\nu) + \frac{c_2(1-\nu)}{r^2}\right]$$

Boundary Conditions (two conditions)

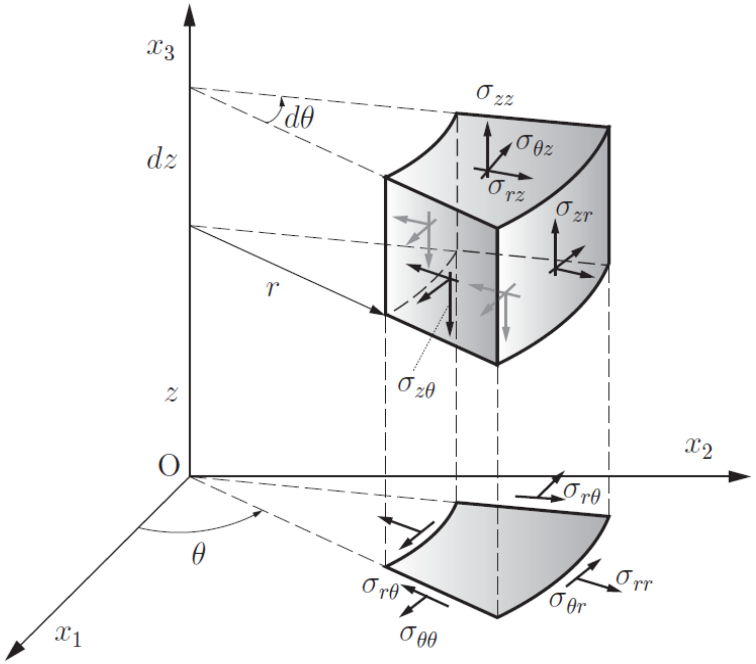
1: To have a finite stress at the origin $c_2 = 0$

2: $\sigma_{rr}\big|_{r=r_i} = 0 \;\Rightarrow\; c_1 = \frac{(1-\nu)\alpha}{r_e^2}\int_{r_i}^{r_e} T(r)rdr$

$$\sigma_{rr} = E\alpha\left[\frac{1}{r_e^2}\int_0^{r_e} T(r)rdr - \frac{1}{r^2}\int_0^{r} T(r)rdr\right]; \quad \sigma_{\theta\theta} = E\alpha\left[-T(r) + \frac{1}{r_e^2}\int_0^{r_e} T(r)rdr + \frac{1}{r^2}\int_0^{r} T(r)rdr\right]$$

Mechanics of Solids: Temperature changes

Thermal Stresses: Long Cylinder with fixed ends (plane strain)



STRESS-STRAIN RELATIONS (PLANE STRAIN)

(we need only replace

$$x_1, x_2, x_3 \rightarrow r, \theta, z$$

in the corresponding relations on Cartesian Coordinates)

$$\varepsilon_{rr} = \frac{1}{E}\left(\sigma_{rr} - v\sigma_{\theta\theta} - v\sigma_{zz}\right) + \alpha\Delta T$$

$$\varepsilon_{\theta\theta} = \frac{1}{E}\left(\sigma_{\theta\theta} - v\sigma_{zz} - v\sigma_{rr}\right) + \alpha\Delta T$$

$$\varepsilon_{zz} = \frac{1}{E}\left(\sigma_{zz} - v\sigma_{rr} - v\sigma_{\theta\theta}\right) + \alpha\Delta T$$

For plane strain $\varepsilon_{zz} = 0$

and $\sigma_{zz} = v(\sigma_{rr} + \sigma_{\theta\theta}) - \alpha E\Delta T$

Introduce σ_{zz} into the deformation expressions to obtain

$$\varepsilon_{rr} = \frac{1+v}{E}\left[(1-v)\sigma_{rr} - v\sigma_{\theta\theta}) + \alpha E\Delta T\right]$$

$$\varepsilon_{\theta\theta} = \frac{1+v}{E}\left[(1-v)\sigma_{\theta\theta} - v\sigma_{rr}) + \alpha E\Delta T\right]$$

Mechanics of Solids: Theory of Elasticity; Eqs in cylindrical coordinates

Thermal Stresses: Long Cylinder with fixed ends (plane strain)

Equation of Equilibrium

$$\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0$$

Stain-Displacement Relations

$$\varepsilon_{rr} = \frac{du_r}{dr}; \quad \varepsilon_{\theta\theta} = \frac{u_r}{r}$$

Compatibility Relation

$$\frac{d(r\varepsilon_{\theta\theta})}{dr} - \varepsilon_{rr} = 0$$

$$\Rightarrow \quad r\frac{d\varepsilon_{\theta\theta}}{dr} + \varepsilon_{\theta\theta} - \varepsilon_{rr} = 0$$

The problem can be solved by introducing a stress function:

Define the stress function ϕ such that:

$$\sigma_{rr} = \frac{\phi(r)}{r}; \quad \sigma_{\theta\theta} = \frac{d\phi(r)}{dr}$$

The equations of equilibrium are satisfied with these stresses.
Introduce these stresses in the stress-strain relations

$$\varepsilon_{rr} = \frac{1+\nu}{E}\left[(1-\nu)\sigma_{rr} - \nu\sigma_{\theta\theta} + \alpha E\Delta T\right]$$

$$\varepsilon_{\theta\theta} = \frac{1+\nu}{E}\left[(1-\nu)\sigma_{\theta\theta} - \nu\sigma_{rr} + \alpha E\Delta T\right]$$

And the resulting strains in the compatibility equation
with $\Delta T \rightarrow T(r)$

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d(r\phi)}{dr}\right] = -\frac{2\alpha E}{1-\nu}\frac{dT(r)}{dr}$$

Mechanics of Solids: Theory of Elasticity; Eqs in cylindrical coordinates

Thermal Stresses Long Cylinder with fixed ends (plane strain)

Integrate the equation

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d(r\phi)}{dr}\right] = -\frac{2\alpha E}{1-\nu}\frac{dT(r)}{dr}$$

$$\phi = -\frac{\alpha E}{1-\nu}\frac{1}{r}\int_{r_i}^{r} T(r)rdr + \frac{C_1 r}{2} + \frac{C_2}{r}$$

Use the result to obtain stresses $\sigma_{rr} = \frac{\phi(r)}{r}; \quad \sigma_{\theta\theta} = \frac{d\phi(r)}{dr}$

SOLID CYLINDER:

Boundary conditions

1. $C_2 = 0$ for finite stresses at the origin

2. zero stresses at $\sigma_{rr}\big|_{r=r_e} = 0$

$$\boxed{C_1 = \frac{2E\alpha}{1-\nu}\frac{1}{r_e^2}\int_0^{r_e} T(r)rdr}$$

$$\sigma_{rr} = \frac{E\alpha}{1-\nu}\left[\frac{1}{r_e^2}\int_0^{r_e} T(r)rdr - \frac{1}{r^2}\int_0^{r} T(r)rdr\right];$$

$$\sigma_{\theta\theta} = \frac{E\alpha}{1-\nu}\left[-T(r) + \frac{1}{r_e^2}\int_0^{r_e} T(r)rdr + \frac{1}{r^2}\int_0^{r} T(r)rdr\right]$$

$$\sigma_{zz} = \frac{E\alpha}{1-\nu}\left[\frac{2\nu}{r_e^2}\int_0^{r_e} T(r)rdr - T(r)\right]$$

Mechanics of Solids: Theory of Elasticity; Eqs in cylindrical coordinates

Thermal Stresses Long Cylinder with fixed ends (plane strain)

Integrate the equation

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d(r\phi)}{dr}\right] = -\frac{2\alpha E}{1-\nu}\frac{dT(r)}{dr}$$

$$\phi = -\frac{\alpha E}{1-\nu}\frac{1}{r}\int_{r_i}^{r} T(r)r\,dr + \frac{C_1 r}{2} + \frac{C_2}{r}$$

Use the result to obtain stresses $\sigma_{rr} = \frac{\phi}{r}; \quad \sigma_{\theta\theta} = \frac{d\phi}{dr}$

HOLLOW CYLINDER:

Boundary conditions

zero stresses at $\sigma_{rr}\big|_{r=r_i} = \sigma_{rr}\big|_{r=r_e} = 0$

$$C_1 = \frac{2E\alpha}{1-\nu}\frac{1}{r_e^2 - r_i^2}\int_{r_i}^{r_e} T(r)r\,dr$$

$$C_2 = -\frac{2E\alpha}{1-\nu}\frac{r_i^2}{r_e^2 - r_i^2}\int_{r_i}^{r_e} T(r)r\,dr$$

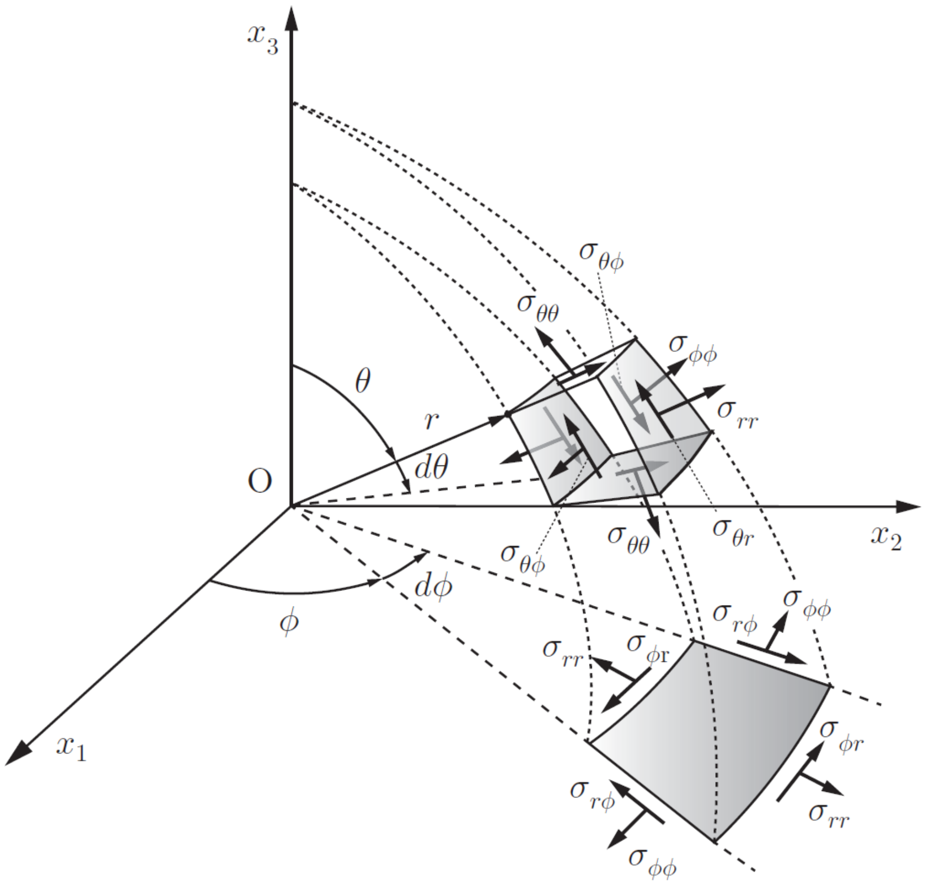
$$\sigma_{rr} = \frac{E\alpha}{1-\nu}\frac{1}{r^2}\left[\frac{r^2 - r_i^2}{r_e^2 - r_i^2}\int_{r_i}^{r_e} T(r)r\,dr - \int_{r_i}^{r} T(r)r\,dr\right]$$

$$\sigma_{\theta\theta} = \frac{E\alpha}{1-\nu}\frac{1}{r^2}\left[\frac{r^2 + r_i^2}{r_e^2 - r_i^2}\int_{r_i}^{r_e} T(r)r\,dr + \int_{r_i}^{r} T(r)r\,dr - T(r)r^2\right]$$

$$\sigma_{zz} = \frac{E\alpha}{1-\nu}\frac{2\nu}{r_e^2 - r_i^2}\left[\int_{r_i}^{r_e} T(r)r\,dr - T(r)\right]$$

Mechanics of Solids: Theory of Elasticity-Eqs in spherical coordinates

THERMAL STRESSES IN SPHERES



Stress Components

$$\boxed{\sigma_{\theta\theta} = \sigma_{\varphi\varphi}; \quad \sigma_{RR}}$$

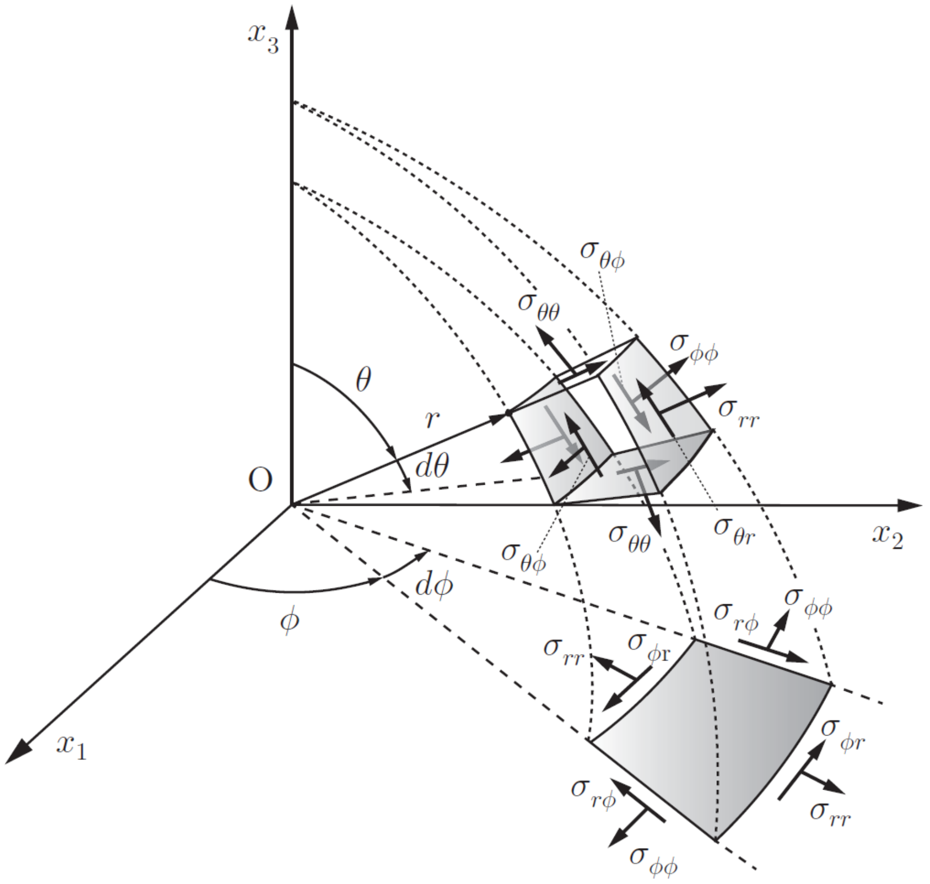
Only one equation of equilibrium

$$\boxed{\frac{\partial \sigma_{RR}}{\partial R} + \frac{2}{R}(\sigma_{RR} - \sigma_{\theta\theta}) = 0}$$

Strain-displacement relations:

$$\boxed{\varepsilon_{RR} = \frac{\partial u_R}{\partial R}; \quad \varepsilon_{\theta\theta} = \varepsilon_{\varphi\varphi} = \frac{u_R}{R}}$$

THERMAL STRESSES IN SPHERES



Stress-Strain Relations

$$\sigma_{RR} = \frac{E}{(1+v)(1-2v)}\left[(1-v)\varepsilon_{RR} + v\varepsilon_{\theta\theta} + v\varepsilon_{\varphi\varphi}\right] - \frac{E}{(1-2v)}\alpha\Delta T$$

$$\sigma_{\theta\theta} = \frac{E}{(1+v)(1-2v)}\left[(1-v)\varepsilon_{\theta\theta} + v\varepsilon_{RR} + v\varepsilon_{\varphi\varphi}\right] - \frac{E}{(1-2v)}\alpha\Delta T$$

Replace the strains with

$$\varepsilon_{RR} = \frac{\partial u_R}{\partial R}; \qquad \varepsilon_{\theta\theta} = \varepsilon_{\varphi\varphi} = \frac{u_R}{R}$$

Introduce the results in equilibrium eq.

$$\frac{d}{dR}\left[\frac{1}{R^2}\frac{d}{dR}\left(R^2 u_R\right)\right] = \frac{1+v}{1-v}\alpha\frac{dT(R)}{dR}$$

One equation with one unknown

Thermal Stresses in Spheres

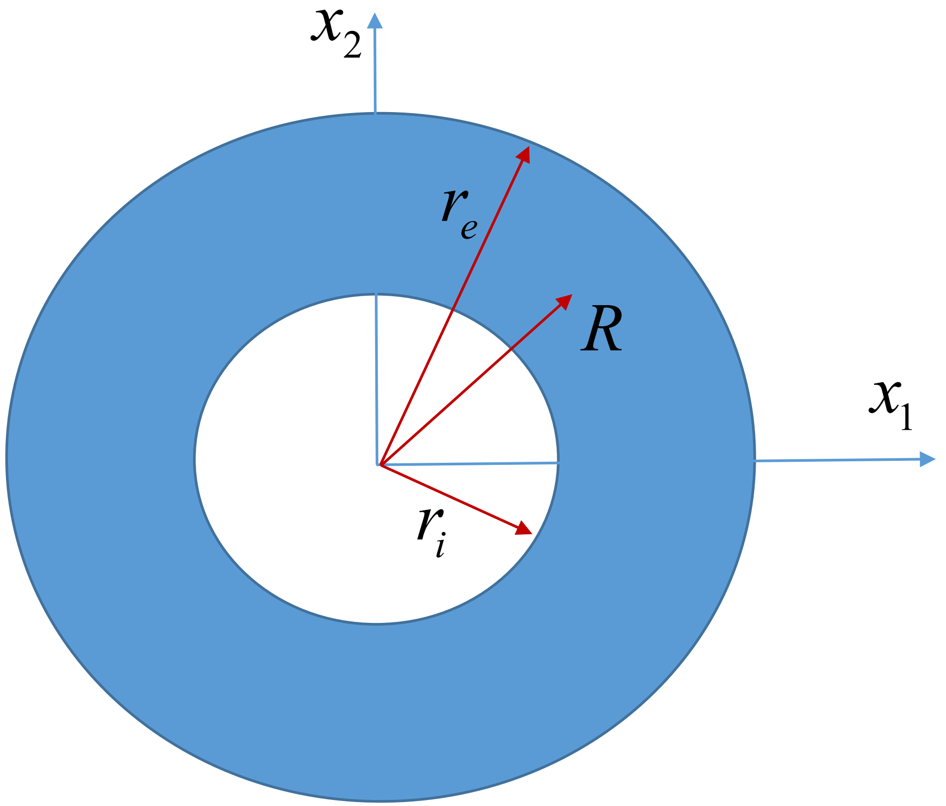
$$\frac{d}{dR}\left[\frac{1}{R^2}\frac{d}{dR}\left(R^2 u_R\right)\right]=\frac{1+\nu}{1-\nu}\alpha\frac{dT(R)}{dR}$$

Solution of this equation is

$$u_R=\frac{1+\nu}{1-\nu}\frac{\alpha}{R^2}\int_{r_i}^{R}T(R)R^2dR+C_1R+\frac{C_2}{R^2}$$

In a solid sphere $r_i=0$

In a hollow sphere r_i is the internal radius



THERMAL STRESSES IN SPHERES

With the solution for the displacement

$$u_R = \frac{1+\nu}{1-\nu}\frac{\alpha}{R^2}\int_{r_i}^{R} T(R)R^2 dR + C_1 R + \frac{C_2}{R^2}$$

we calculate the strains

$$\varepsilon_{RR} = \frac{\partial u_R}{\partial R}; \qquad \varepsilon_{\theta\theta} = \varepsilon_{\varphi\varphi} = \frac{u_R}{R}$$

From the stress-strain relation, we calculate the stresses

$$\sigma_{RR} = -\frac{2E\alpha}{1-\nu}\frac{1}{R^3}\left[\int_{r_i}^{R} T(R)R^2 dR\right] + C_1\frac{E}{1-2\nu} - C_2\frac{2E}{(1+\nu)R^3}$$

$$\sigma_{\theta\theta} = \frac{2E\alpha}{1-\nu}\frac{1}{R^3}\left[\int_{r_i}^{R} T(R)R^2 dR\right] + C_1\frac{E}{1-2\nu} + C_2\frac{2E}{(1+\nu)R^3} - \frac{E\alpha T(R)}{1-\nu}$$

THERMAL STRESSES IN SPHERES

Solid Sphere

$$\sigma_{RR} = -\frac{2E\alpha}{1-v}\frac{1}{R^3}\left[\int_{r_i}^{R} T(R)R^2 dR\right] + C_1\frac{E}{1-2v} - C_2\frac{2E}{(1+v)R^3}$$

$$\sigma_{\theta\theta} = \frac{2E\alpha}{1-v}\frac{1}{R^3}\left[\int_{r_i}^{R} T(R)R^2 dR\right] + C_1\frac{E}{1-2v} + C_2\frac{2E}{(1+v)R^3} - \frac{E\alpha T(R)}{1-v}$$

Boundary Conditions

1. The stresses must be finite at the origin $\Rightarrow C_2 = 0$

2. Free external surface $\sigma_{RR}\big|_{R=r_e} = 0$

$$\Rightarrow C_1 = \frac{2\alpha(1-2v)}{1-v}\frac{1}{R^3}\int_0^{r_e} T(R)R^2 dR$$

$$\sigma_{RR} = \frac{2E\alpha}{1-v}\left[\frac{1}{r_e^3}\int_0^{r_e} T(R)R^2 dR - \frac{1}{R^3}\int_0^{r_e} T(R)R^2 dR\right]$$

$$\sigma_{\theta\theta} = \frac{E\alpha}{1-v}\left[\frac{1}{r_e^3}\int_0^{r_e} T(R)R^2 dR + \frac{1}{R^3}\int_0^{r_e} T(R)R^2 dR - T(R)\right]$$

THERMAL STRESSES IN SPHERES

Hollow Sphere

$$\boxed{\begin{aligned}\sigma_{RR} &= -\frac{2E\alpha}{1-v}\frac{1}{R^3}\left[\int_{r_i}^{R} T(R)R^2 dR\right] + C_1\frac{E}{1-2v} - C_2\frac{2E}{(1+v)R^3} \\ \sigma_{\theta\theta} &= \frac{2E\alpha}{1-v}\frac{1}{R^3}\left[\int_{r_i}^{R} T(R)R^2 dR\right] + C_1\frac{E}{1-2v} + C_2\frac{2E}{(1+v)R^3} - \frac{E\alpha T(R)}{1-v}\end{aligned}}$$

Boundary Conditions

Two conditions

$$\sigma_{RR}\big|_{R=r_i} = \sigma_{RR}\big|_{R=r_e} = 0$$

$$\boxed{\begin{aligned}&\frac{EC_1}{1-2v} - \frac{2EC_2}{1+v}\frac{1}{r_i^3} = 0 \\ &-\frac{2E\alpha}{1-v}\frac{1}{r_e^3}\int_{r_i}^{r_e} T(R)R^2 dR + \frac{EC_1}{1-2v} - \frac{2EC_2}{1+v}\frac{1}{r_e^3} = 0\end{aligned}}$$

Solve the system for the two constants

Mechanics of Solids: Theory of Elasticity-Eqs in spherical coordinates

THERMAL STRESSES IN SPHERES

Hollow Sphere

$$\sigma_{RR} = -\frac{2E\alpha}{1-v}\frac{1}{R^3}\left[\int_{r_i}^{R} T(R)R^2 dR\right] + C_1\frac{E}{1-2v} - C_2\frac{2E}{(1+v)R^3}$$

$$\sigma_{\theta\theta} = \frac{2E\alpha}{1-v}\frac{1}{R^3}\left[\int_{r_i}^{R} T(R)R^2 dR\right] + C_1\frac{E}{1-2v} + C_2\frac{2E}{(1+v)R^3} - \frac{E\alpha T(R)}{1-v}$$

$$\Downarrow$$

$$\sigma_{RR} = \frac{2E\alpha}{1-v}\left[\frac{R^3 - r_i^3}{(r_e^3 - r_i^3)}\frac{1}{R^3}\int_{r_i}^{r_e} T(R)R^2 dR - \frac{1}{R^3}\int_{r_i}^{r_e} T(R)R^2 dR\right]$$

$$\sigma_{\theta\theta} = \frac{2E\alpha}{1-v}\left[\frac{2R^3 + r_i^3}{2(r_e^3 - r_i^3)}\frac{1}{R^3}\int_{r_i}^{r_e} T(R)R^2 dR + \frac{1}{2R^3}\int_{r_i}^{r_e} T(R)R^2 dR - \frac{T(R)}{2}\right]$$

THERMAL STRESSES IN CYLINDERS AND SPHERES

When the temperature at the inner and outer surfaces are given we need to solve the Fourier's Law for heat conduction in order to obtain the temperature distribution as a function of the radial coordinate.

This temperature distribution is inserted into the corresponding equations to obtain the resulting thermal stresses.